8.4.1 Exercises

For a link to all of the additional resources available for this section, click OSttS Chapter 8 materials.

In Exercises 1 - 8, find the inverse of the matrix or state that the matrix is not invertible.

For help with these exercises, click on the resource below:

- Finding the inverse of a matrix using Gauss-Jordan Elimination

1. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

2. $B = \begin{bmatrix} 12 & -7 \\ -5 & 3 \end{bmatrix}$

3. $C = \begin{bmatrix} 6 & 15 \\ 14 & 35 \end{bmatrix}$

4. $D = \begin{bmatrix} 2 & -1 \\ 16 & -9 \end{bmatrix}$

5. $E = \begin{bmatrix} 3 & 0 & 4 \\ 2 & -1 & 3 \\ -3 & 2 & -5 \end{bmatrix}$

6. $F = \begin{bmatrix} 4 & 6 & -3 \\ 3 & 4 & -3 \\ 1 & 2 & 6 \end{bmatrix}$

7. $G = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 11 \\ 3 & 4 & 19 \end{bmatrix}$

8. $H = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 2 & -2 & 8 & 7 \\ -5 & 0 & 16 & 0 \\ 1 & 0 & 4 & 1 \end{bmatrix}$

In Exercises 9 - 11, use one matrix inverse to solve the following systems of linear equations.

For help with these exercises, click on the resource below:

- Finding the inverse of a matrix using Gauss-Jordan Elimination

9. $\begin{cases} 3x + 7y = 26 \\ 5x + 12y = 39 \end{cases}$

10. $\begin{cases} 3x + 7y = 0 \\ 5x + 12y = -1 \end{cases}$

11. $\begin{cases} 3x + 7y = -7 \\ 5x + 12y = 5 \end{cases}$

In Exercises 12 - 14, use the inverse of E from Exercise 5 above to solve the following systems of linear equations.

12. $\begin{cases} 3x + 4z = 1 \\ 2x - y + 3z = 0 \\ -3x + 2y - 5z = 0 \end{cases}$

13. $\begin{cases} 3x + 4z = 0 \\ 2x - y + 3z = 1 \\ -3x + 2y - 5z = 0 \end{cases}$

14. $\begin{cases} 3x + 4z = 0 \\ 2x - y + 3z = 0 \\ -3x + 2y - 5z = 1 \end{cases}$

15. This exercise is a continuation of Example 8.3.3 in Section 8.3 and gives another application of matrix inverses. Recall that given the position matrix P for a point in the plane, the matrix RP corresponds to a point rotated 45° counterclockwise from P where

$$R = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

(a) Find R^{-1}.

(b) If RP rotates a point counterclockwise 45°, what should $R^{-1}P$ do? Check your answer by finding $R^{-1}P$ for various points on the coordinate axes and the lines $y = \pm x$.

(c) Find $R^{-1}P$ where P corresponds to a generic point $P(x, y)$. Verify that this takes points on the curve $y = \frac{2}{x}$ to points on the curve $x^2 - y^2 = 4$.

16. A Sasquatch's diet consists of three primary foods: Ippizuti Fish, Misty Mushrooms, and Sun Berries. Each serving of Ippizuti Fish is 500 calories, contains 40 grams of protein, and has no Vitamin X. Each serving of Misty Mushrooms is 50 calories, contains 1 gram of protein, and 5 milligrams of Vitamin X. Finally, each serving of Sun Berries is 80 calories, contains no protein, but has 15 milligrams of Vitamin X.[9]

    (a) If an adult male Sasquatch requires 3200 calories, 130 grams of protein, and 275 milligrams of Vitamin X daily, use a matrix inverse to find how many servings each of Ippizuti Fish, Misty Mushrooms, and Sun Berries he needs to eat each day.

    (b) An adult female Sasquatch requires 3100 calories, 120 grams of protein, and 300 milligrams of Vitamin X daily. Use the matrix inverse you found in part (a) to find how many servings each of Ippizuti Fish, Misty Mushrooms, and Sun Berries she needs to eat each day.

    (c) An adolescent Sasquatch requires 5000 calories, 400 grams of protein daily, but no Vitamin X daily.[10] Use the matrix inverse you found in part (a) to find how many servings each of Ippizuti Fish, Misty Mushrooms, and Sun Berries she needs to eat each day.

17. Matrices can be used in cryptography. Suppose we wish to encode the message 'BIGFOOT LIVES'. We start by assigning a number to each letter of the alphabet, say $A = 1$, $B = 2$ and so on. We reserve 0 to act as a space. Hence, our message 'BIGFOOT LIVES' corresponds to the string of numbers '2, 9, 7, 6, 15, 15, 20, 0, 12, 9, 22, 5, 19.' To encode this message, we use an invertible matrix. Any invertible matrix will do, but for this exercise, we choose

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 1 & -2 \\ -7 & 1 & -1 \end{bmatrix}$$

    Since A is 3×3 matrix, we encode our message string into a matrix M with 3 rows. To do this, we take the first three numbers, 2 9 7, and make them our first column, the next three numbers, 6 15 15, and make them our second column, and so on. We put 0's to round out the matrix.

$$M = \begin{bmatrix} 2 & 6 & 20 & 9 & 19 \\ 9 & 15 & 0 & 22 & 0 \\ 7 & 15 & 12 & 5 & 0 \end{bmatrix}$$

[9] Misty Mushrooms and Sun Berries are the only known fictional sources of Vitamin X.

[10] Vitamin X is needed to sustain Sasquatch longevity only.

To encode the message, we find the product AM

$$AM = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 1 & -2 \\ -7 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 20 & 9 & 19 \\ 9 & 15 & 0 & 22 & 0 \\ 7 & 15 & 12 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 12 & 42 & 100 & -23 & 38 \\ 1 & 3 & 36 & 39 & 57 \\ -12 & -42 & -152 & -46 & -133 \end{bmatrix}$$

So our coded message is '12, 1, −12, 42, 3, −42, 100, 36, −152, −23, 39, −46, 38, 57, −133.' To decode this message, we start with this string of numbers, construct a message matrix as we did earlier (we should get the matrix AM again) and then multiply by A^{-1}.

(a) Find A^{-1}.

(b) Use A^{-1} to decode the message and check this method actually works.

(c) Decode the message '14, 37, −76, 128, 21, −151, 31, 65, −140'

(d) Choose another invertible matrix and encode and decode your own messages.

18. Using the matrices A from Exercise 1, B from Exercise 2 and D from Exercise 4, show $AB = D$ and $D^{-1} = B^{-1}A^{-1}$. That is, show that $(AB)^{-1} = B^{-1}A^{-1}$.

19. Let M and N be invertible $n \times n$ matrices. Show that $(MN)^{-1} = N^{-1}M^{-1}$ and compare your work to Exercise 31 in Section 5.2.

Checkpoint Quiz 8.4

1. Consider the system: $\begin{cases} 3x + 2y + z = 1 \\ x - y + 3z = -3 \\ 2x + 5y - 11z = 8 \end{cases}$

(a) Solve this system using a matrix inverse.

(b) This is the same system that was in Checkpoint Quiz 8.2. Compare the row operations you used to solve the system using an augmented matrix and the row operations you used to determine the inverse of the coefficient matrix of this system.

For worked out solutions to this quiz, click the links below:

- Quiz Solution Part 1
- Quiz Solution Part 2
- Quiz Solution Part 3

8.4.2 ANSWERS

1. $A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

2. $B^{-1} = \begin{bmatrix} 3 & 7 \\ 5 & 12 \end{bmatrix}$

3. C is not invertible

4. $D^{-1} = \begin{bmatrix} \frac{9}{2} & -\frac{1}{2} \\ 8 & -1 \end{bmatrix}$

5. $E^{-1} = \begin{bmatrix} -1 & 8 & 4 \\ 1 & -3 & -1 \\ 1 & -6 & -3 \end{bmatrix}$

6. $F^{-1} = \begin{bmatrix} -\frac{5}{2} & \frac{7}{2} & \frac{1}{2} \\ \frac{7}{4} & -\frac{9}{4} & -\frac{1}{4} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$

7. G is not invertible

8. $H^{-1} = \begin{bmatrix} 16 & 0 & 3 & 0 \\ -90 & -\frac{1}{2} & -\frac{35}{2} & \frac{7}{2} \\ 5 & 0 & 1 & 0 \\ -36 & 0 & -7 & 1 \end{bmatrix}$

The coefficient matrix is B^{-1} from Exercise 2 above so the inverse we need is $(B^{-1})^{-1} = B$.

9. $\begin{bmatrix} 12 & -7 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 26 \\ 39 \end{bmatrix} = \begin{bmatrix} 39 \\ -13 \end{bmatrix}$ So $x = 39$ and $y = -13$.

10. $\begin{bmatrix} 12 & -7 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$ So $x = 7$ and $y = -3$.

11. $\begin{bmatrix} 12 & -7 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} -7 \\ 5 \end{bmatrix} = \begin{bmatrix} -119 \\ 50 \end{bmatrix}$ So $x = -119$ and $y = 50$.

The coefficient matrix is $E = \begin{bmatrix} 3 & 0 & 4 \\ 2 & -1 & 3 \\ -3 & 2 & -5 \end{bmatrix}$ from Exercise 5, so $E^{-1} = \begin{bmatrix} -1 & 8 & 4 \\ 1 & -3 & -1 \\ 1 & -6 & -3 \end{bmatrix}$

12. $\begin{bmatrix} -1 & 8 & 4 \\ 1 & -3 & -1 \\ 1 & -6 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ So $x = -1$, $y = 1$ and $z = 1$.

13. $\begin{bmatrix} -1 & 8 & 4 \\ 1 & -3 & -1 \\ 1 & -6 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \\ -6 \end{bmatrix}$ So $x = 8$, $y = -3$ and $z = -6$.

14. $\begin{bmatrix} -1 & 8 & 4 \\ 1 & -3 & -1 \\ 1 & -6 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix}$ So $x = 4$, $y = -1$ and $z = -3$.

16. (a) The adult male Sasquatch needs: 3 servings of Ippizuti Fish, 10 servings of Misty Mushrooms, and 15 servings of Sun Berries daily.

    (b) The adult female Sasquatch needs: 3 servings of Ippizuti Fish and 20 servings of Sun Berries daily. (No Misty Mushrooms are needed!)

    (c) The adolescent Sasquatch requires 10 servings of Ippizuti Fish daily. (No Misty Mushrooms or Sun Berries are needed!)

17. (a) $A^{-1} = \begin{bmatrix} 1 & 2 & 1 \\ 17 & 33 & 19 \\ 10 & 19 & 11 \end{bmatrix}$

    (b) $\begin{bmatrix} 1 & 2 & 1 \\ 17 & 33 & 19 \\ 10 & 19 & 11 \end{bmatrix} \begin{bmatrix} 12 & 42 & 100 & -23 & 38 \\ 1 & 3 & 36 & 39 & 57 \\ -12 & -42 & -152 & -46 & -133 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 20 & 9 & 19 \\ 9 & 15 & 0 & 22 & 0 \\ 7 & 15 & 12 & 5 & 0 \end{bmatrix}$ ✓

    (c) 'LOGS RULE'